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Heteropoiesis: The Systemic Fecundity of Impure Methods

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Abstract. Based on the controversies of the late 19th and early 20th centuries concerning the foundations of mathematics, we discuss the significance, beyond the field of mathematics, of methodological impurities. In other words, we highlight instances where rationality is extended through excursions outside of a basic language which, after a detour, return to the vicinity of the question posed in the initial language. We find these concerns particularly evident in the work of Hermann Weyl and Niels Bohr regarding quantum physics.

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When I observe my colleagues in seminars as they practise mathematics and discuss their research with one another, I get the strong impression that they inhabit a realm where a certain language is shared by the whole group, the semantics of which are obvious and crystal clear to everyone. There is, for example, a language of probability theory through which one navigates familiar territory, where certain established results are common knowledge, concerning processes, ergodic theory, stochastic analysis, and so on. In practice, this usage draws on celebrated treatises that have synthesised the important knowledge on these questions. Everything suggests that there is a well-understood "dialect" and that what is of interest to research concerns the questions formulated within this language, which most often do not call into question the methodological framework, whether for the group during presentations or in even the most imaginative individual work. To such an extent that mathematicians often attach little importance to new concepts, even if they possess particular properties, as long as they do not yield conclusions that shed light on already known objects. Given the relatively easy proliferation of ontological enrichments now that mathematics has been formalised, the criterion of providing answers to the natural questions arising from the corpus becomes important and even paramount. This ties in with an ancient custom, both ethical and aesthetic, most often accepted as a convention of politeness.

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§ 1. — The principle of purity of methods.

This issue has become a much-debated topic in the philosophy of mathematics, involving subtle semantic nuances which we do not absolutely need to consider here.[(1)] Historically, this was a more or less explicit rule of the craft. It was considered that it was better, more elegant and more convincing to prove a theorem using only the concepts referred to in the statement, or those essential for understanding the statement. Until the end of the 19th century, mathematics was not entirely formalised. Its boundaries were not really clear. D'Alembert had sensed this lack of clarity and, in the context of mechanics and its applications, spoke of "mixed" mathematics. In practical terms, what did mathematics consist of? It was

[(1)] For further reading, see A. Arana, *Elements of Purity*, Cambridge University Press, 2025.

the language used in the great treatises by the great mathematicians (Euclid, Descartes, Euler, etc.). As its boundaries were not clearly defined, there was a tendency to be sparing with assumptions.

This principle was still held in such high regard at the start of the 20th century that David Hilbert adopted an attitude which, from our perspective today, might seem strange.[2]

In 1899, in his remarkable treatise *Grundlagen der Geometrie*, in which he examined the dependence and independence of the various axioms of geometry — a work that made a strong impression on his contemporaries — he had, in particular, clarified Desargues's curious statement. This is a subtle assertion in plane geometry which becomes almost self-evident if one thinks of the figure as the drawing of a tetrahedron in space. Hilbert explained this phenomenon by showing that Desargues's property was false for certain plane geometries other than the usual Euclidean geometry. These "non-Desarguesian" geometries are not projections of three-dimensional Euclidean geometry. Continuing his investigation into the foundations of mathematics, in 1900 Hilbert announced, in the introduction to his 23 problems formulated at the International Congress of Mathematicians in Paris, a famous research *programme* to establish the consistency of arithmetic, based on the idea that any clear question about mathematics must have a positive or negative answer, and — importantly — the method he advocated consisted of applying the principle of purity of methods, starting from elementary formulas, to show that detours via abstract notions could be reduced to direct paths without straying beyond the elementary formulas.

However, as is now well known, the idea that there was no question without an answer has proved to be incorrect. Certain statements in arithmetic are undecidable. And Hilbert's programme ended in failure. Detours are creative in mathematics. To put it figuratively, one might say that Desargues's assertion is typical of mathematics in general. One can prove more things using impure methods. Taking detours is fecund. One obtains more results on real numbers by going via complex numbers, and more on integers by going via set theory. Given a statement, there is no algorithm to determine whether it is provable or not; one must try.

[2] Cf. P. Mancosu, *Infini, logique, géométrie*, Vrin, 2015, p. 373 *et seq*.

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Fermat's Last Theorem was proved by Andrew Wiles using algebraic geometry.[3]

§ 2. — Ignorabimus.

A few decades before Hilbert took his stand at the Paris Congress, the German physiologist Emil du Bois-Reymond, in a speech at the 45th Congress of Naturalists and Physicists in Leipzig in 1872, had defended the idea that science, even as it gradually expands our knowledge, necessarily leaves two major questions unresolved. Firstly, the ultimate nature of matter and force. Secondly, the relationship between the material or physical states of an organism, on the one hand, and its consciousness (thoughts, feelings, desires), on the other. He concluded his remarks with the famous phrase *Ignoramus et ignorabimus*, emphasising that, of course, there are things we do not know, but also things we will never know.

Emil du Bois-Reymond's viewpoint belonged to a current of thinkers that persisted beyond the 19th century, including Émile Boutroux, Henri Bergson and many others, who considered that the abandonment of religious explanations did not in itself guarantee that science would explain everything. Faced with the progressive movements of the 19th century — whether materialist, positivist or Saint-Simonian — they rejected scientism, which preserved religious faith by transferring it to science. In today's terms, they emphasised the possibility of permanent *incompleteness*.

This position itself stemmed from an earlier critique of Leibniz's *principle of sufficient reason*, which can be stated as follows:

> "*Nothing ever comes to pass without there being a cause or at least a reason determining it, that is, something to give an a priori reason why it is existent rather than non-existent, and in this wise rather than in any other.*"[4]

[3] In several cases, by clearly distinguishing between the pure language and that of the extension, it can be proved that the result cannot be obtained without a detour, cf. G. Longo, "Reflections on Concrete Incompleteness", *Philosophia Mathematica*, vol. 19, no. 3, 2011, p. 255-280. It is known, for example, that certain undecidable statements in arithmetic are provable in set theory with large cardinal axioms.

[4] G. W. Leibniz, *Theodicy*, ed. A. Farrer, transl. E. M. Huggard, La Salle (Ill.), Open Court, 1985 [1951], Part I, § 44, p. 147; French original in C. I. Gerhardt (ed.), *Die philosophischen Schriften von Gottfried Wilhelm Leibniz*, vol. VI, Berlin, Weidmann, 1885, p. 127.

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It is summed up by the phrase *nihil est sine ratione*: nothing is without a reason. It is the assertion that the universe of causes is rich enough to account for all that exists. Such a principle could have been formulated by an 18th-century mathematician who, on the one hand, envisaged that the world could be represented by mathematics, and on the other hand, believed that mathematics was *complete* — as was generally believed until the early 20th century. This line of thought underestimates the complexity of combinatorics. By connecting with one another, monads form complex entities. However, conversely, the question of whether a given complex entity is accessible through combinations is one that, at the time, no one considered to be without an answer, since the principle of sufficient reason guarantees an answer and conceals this difficulty.

The principle of purity of methods derives from the principle of sufficient reason; both condemn the use of *ad hoc* explanations and criticise Newton's action at a distance as an appeal to *occult forces*. These two principles are pre-modern forms of materialism. Leibniz, as a good Protestant, attributes to God the knowledge and power to create the best of all possible worlds and to fill in all the causes that elude us, provided we have faith.

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§ 3. — Heteropoiesis.

The results of the 1930s by Gödel, Church and Turing on *incompleteness* refuted Hilbert's programme. But — and this is the central point — they made room for an *ignorabimus* based on arguments entirely different from those of du Bois-Reymond or Bergson, and which neither had envisaged. For what emerges, what is radically new, is the fact that the combinatorics of signs, by virtue of its own complexity, is sufficient to yield undecidability and incompleteness.

To present this novelty as simply as possible, one might say that the discovery lies in the fact that deduction, when extended across a language broader than the usual language, makes it possible to obtain new statements within the usual language itself. We shall refer to this *principle of impurity of methods* by the term *heteropoiesis*,

to signify the fecundity of heterodox creation through detours and excursions.[5]

Let us return to Hilbert for a moment. It is quite striking that this prodigious mathematician and logician, together with his collaborators, pursued the search for a proof of the consistency of arithmetic for 30 years, in accordance with the programme he had announced. So much so that he publicly reiterated this motivation in the very year that Gödel proved it to be impossible. Hilbert read out on the radio an abridged version of the end of his speech of 8 September 1930 at the annual meeting of the German Society of Scientists and Physicians:

> *We must not believe those, who today with philosophical bearing and a tone of superiority prophesy the downfall of culture and accept the ignorabimus. For us there is no ignorabimus, and in my opinion even none whatever in natural science. In place of the foolish ignorabimus let stand our slogan:*
>
> *We must know,*
> *We will know.*[6]

Hilbert does not treat the consistency of arithmetic as a conjecture that could be disproved or confirmed by the talents of mathematicians. This is probably because, as a conjecture, if it were refuted, the elementary truths of arithmetic would be imperilled. It is not impossible that the results of Jacques Herbrand — who worked in contact with Hilbert and his disciples, and who had just proved the consistency of a part of arithmetic — may have influenced Hilbert in adopting his radical stance.

Finally, following the incompleteness results of the 1930s, does this place us on Emil du Bois-Reymond's side? Has this physiologist's intuition simply been confirmed, with the two enigmas of rationalism that he presents being confirmed as permanent ignorance? No, because du Bois-Reymond's argument is purely

[5] Let us mention the work of Alessandro Sarti *et al.*, who develop the related concept of heterogenesis, cf. A. Sarti, G. Citti, D. Piotrowski, *Differential Heterogenesis*, Springer, 2022.

[6] D. Hilbert, "Naturerkennen und Logik", Königsberg address of 8 September 1930, *Die Naturwissenschaften*, vol. 18, 1930, p. 959-963. The passage follows the abridged radio version, transcribed and translated by J. T. Smith, "David Hilbert's Radio Address", *Loci: Convergence*, vol. 11, 2014, p. 2-3; the full printed address is translated by W. Ewald, *From Kant to Hilbert*, vol. 2, Oxford University Press, 1996, p. 1157-1165.

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philosophical, whereas a new order of causality has emerged which proves to be creative by its very nature. We have moved on to a more concrete reality. This new form of reason is *combinatorics*. It compels us to view all fields of knowledge in which combinations of signs are present in a different light — that is to say, mathematics and mathematised sciences, chemistry, and the biochemistry of open systems. Even in philosophy, it suggests a new reading of Plato's *Allegory of the Cave*, to which Heidegger devoted a famous lecture course. This account, dating from the early 4th century BC, is probably the most widely commented-upon parable in philosophy; it bears witness to the extraordinary centre of intellectual activity that was Athens.

Heidegger immediately dismisses any approach that does not begin with a detailed analysis of the original Greek text.[7] As a distinguished Hellenist, this was his way of taking to the lectern for his lecture course in Freiburg in 1931-32. But it is also a way of drawing us into his own phraseology. A summary of Plato's account will suffice:

Men are chained in a cave facing a wall on which they see moving shapes — the shadows of people carrying objects along a path illuminated from behind by the sun, which the prisoners cannot see. Plato envisages that one of the captives is set free; despite being dazzled, he manages to reach the path in full light, to see what is happening there, and to come back down to inform his companions. But his companions, having already ascribed meanings to what they saw, are reluctant to abandon them in favour of these new doctrines.

What is most interesting for us here is the initial staging of the allegory: the human condition represented by this cave with its shadows, the origin of which is unknown, without even bringing in the freed prisoner who explores and returns to bear witness. This theatrical arrangement alone is by no means trivial. Plato's staging, in which hidden things provide, *through a physical process*, the reality perceived by the captives, is fascinating. This — simple — idea foreshadows what we must take into account twenty-five centuries later. It is not a matter of gods exerting influence over the lives of mortals. Plato is not constructing a religious parable. Nor is it a representation of the process of abstraction towards essence, although Plato does use it for this purpose to support his theory of Ideas. For

[7] Cf. M. Heidegger, *De l'essence de la vérité*, French transl. A. Boutot, Gallimard, 2001, p. 39-40.

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the prisoners perceive only what their senses reveal to them on the wall. Human beings are depicted as witnessing a world in which certain phenomena are concealed, yet these are the direct *material* causes of what is observed. How is this possible? This is where Hermann Weyl sheds further light on the matter.[8]

He considers that the idea of *Dasein* — that is, "a being whose very existence entails an understanding of its own being, a being that is always itself and understands itself as being-in-the-world and being-with" — is relevant to the foundations of mathematics. It is well known that he took part in Brouwer's intuitionist movement[9] and he emphasises that his ideas on this subject predated the publication of *Sein und Zeit*:

> *With our mathematical construction, as I said some years before Heidegger's* Sein und Zeit *appeared, we stand at that intersection of bondage and freedom which is the essence of man himself.*

He offers another allegory, one that rejects both the religious outlook and positivism:

> *But in following the development of science we have seen that no other ground is left for science to build on than this dark but very solid rock which I once more call the concrete* Dasein *of man in his world. So the house stands on firm ground. But it has no roof, its construction reveals that the world is not closed but points beyond itself to what is represented by the symbols but can never be realized on the human level.*[10]

However, Weyl distances himself from the generalised use of *Dasein* and condemns "the profound but hollow evocations in which Heidegger indulges in his latest publications", as well as the influence of Kierkegaard.

[8] Hermann Weyl, a German speaker, found it difficult to switch to English for his own philosophical works: "I find it extremely hard to translate the essential tenets of existential philosophy into plain English, and I shall deliberately flatten Heidegger's terribly involved phrases", "Man and the Foundations of Science" (1949), in *Mind and Nature*, Princeton University Press, 2009, p. 189.

[9] Cf. N. Bouleau, *Introduction à la philosophie des sciences*, Spartacus-IDH, 2017, chap. III, § 2.

[10] H. Weyl, "Man and the Foundations of Science", *op. cit.*, p. 192 — as for the preceding quotation; French transl.: *Philosophie des mathématiques et des sciences de la nature* (*1927-1949*), transl. C. Lobo, MétisPresses, 2017.

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Hermann Weyl had attended Husserl's lectures.[11] Husserl, who was also a mathematician and an assistant to Weierstrass, had turned to philosophy, to which he contributed a colossal body of work: phenomenology. Drawing on a universal and objective conception of human consciousness, Husserl proposed to find in philosophy a rigour of thought capable of truly grounding the sciences, including mathematics. The errors of great thinkers are often the sign of a significant flaw hidden within common thought. In *The Origin of Geometry* (1936) — as Jacques Derrida points out — continuing his quest for rigour, he demands that mathematics be categorical, in the sense of being capable of defining without any ambiguity what it is talking about. Yet mathematical theories — at least those that are not too poor — have several interpretations. This has a famous name in logic: the Löwenheim-Skolem theorem, which indeed unsettles approaches to the world based on "common sense". Other mathematicians, such as Albert Lautman and Jean Cavaillès, have criticised Husserl's nomological epistemology.[12]

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§ 4. — Epistemic fields of heteropoiesis.

Let us return to the scenario of the *Allegory of the Cave*, in which Plato presents Socrates explaining to Glaucon a quasi-theatrical scene with philosophical significance. Heidegger divides the narrative into four stages.

The first is the stage in which the men are prisoners, chained, and can see only shadows on the wall of the cave in which they find themselves, without even being able to turn round. What is evident at this stage are these shadows.

The second stage is a liberation of the man inside the cave. Socrates supposes that the chains have been undone and that the men find themselves freed, yet they remain within the cave. What

[11] The reference to Hermann Weyl in our discussion would deserve to be explored further. It is true that intuitionism, to which he was close, tends to banish excursions rather than encourage them. His work on relativity (*Temps, espace, matière*, 1918) and on gauge theory is more heteropoietic. But it is above all certain remarks he was the first to make on the analogy between logical statements and molecules that appear prescient today (cf. *Philosophie des mathématiques et des sciences de la nature*, *op. cit.*).

[12] J. Cavaillès, *Sur la logique et la théorie de la science* (1942), PUF, 1947; repr. Vrin, 1976.

is certain at this stage is that the man is still in the cave, from which he is unable to escape.

Then comes the third stage, to which Heidegger devotes his longest commentary. He translates the Greek text as follows; it is Socrates who is speaking:

> *And if he were forcibly dragged up the steep and rugged ascent of the cave and not let go till he had been dragged out into the sunlight, would he not experience pain, and so struggle against this? And would he not, as soon as he emerged into the light, his eyes dazzled, be unable to see any of the things he was now told were unhidden?*[13]

Faced with this spectacle, the man will need a period of adjustment, but he ends up growing accustomed to these novelties and, thinking of his companions, will find himself glad of the change.

The fourth stage is the return of the free man into the cave and the discussion of the difficulty with which this new truth gains acceptance.

Socrates has his interlocutor Glaucon share in the choice of scenario. The narrative could have unfolded differently, as Socrates saw fit. The third stage of the narrative is a further supposition, and suggests that the story could have ended at the close of the second stage, which would then have described a definitive situation: the prisoners freed but unable to gain access to the deeper truth. We thus read, *in absentia*, that Plato — through what he has Socrates say — envisages that the *human condition* might well be a freedom founded on apparent truths.

I believe — and I shall support this thesis with a few further arguments — that our condition today, in the 21st century, corresponds to the situation of the second stage. The first stage has been left behind; knowledge has removed most of our highly visible chains, which once seemed irrefutable. And the crucial fact of the post-20th-century era is that we know that we will not know everything. Why? Because there are causes that were once operative and that have been erased. The angel, the prophets, who would express the divine design — we no longer believe in them, for they too cannot know. The impossibility is systemic.

[13] M. Heidegger, *The Essence of Truth: On Plato's Cave Allegory and* Theaetetus (1931-1932), transl. T. Sadler, Continuum, 2002, p. 30; French ed. used by the author: transl. A. Boutot, Gallimard, 2001.

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The positivists were unaware of this impossibility; it is a discovery of the 20th century. André Malraux aptly evokes their vision:

> *Marcellin Berthelot knew very well that what he was contributing could not account for the world. But when he compared what he was contributing with what the conception of the world had been a hundred years earlier, he had the feeling that three or four hundred years of science would succeed in giving humankind knowledge of the world. And it is on this sort of hope that we still live. None of us, in our civilisation, believes that they possess the meaning of the world. But many still believe that they will find it.*[(14)]

We live on this sort of hope — at least the vast majority of people do — and this belief, actively nurtured, is one of the driving forces of the economy.

What we have called the method of *heteropoiesis* — consisting of a discursive journey that escapes from the usual language by mobilising different entities, then finally returns to the vicinity of the point of origin, in a region where the initial usual language is still in force — is a process that is very much present *outside mathematics*. A few examples will show its importance in the history of thought.

The first, which may serve as an archetype, is the mythical narrative of the *Odyssey* itself. After many adventures, Odysseus returns to Ithaca, whence he had set out, where the same court life goes on around Queen Penelope. But he has changed. He returns laden with a vast experience forged in the course of his voyage. At first he is recognised by no one, save his dog.[(15)] We may therefore speak of *Odyssean excursions* for these detours *in an almost closed loop*.

The heteropoietic process resembles the allegory of the cave in its complete form, with its stages three and four. Yet it is different. For, according to Plato, what is seen by the one who has ascended, who has been drawn towards the light, is the unquestionable *truth*, the true in its uniqueness. By contrast, the processes of Odyssean excursion do not provide a spectacle of fundamental causal truths, but the discovery of objects, stones, plants, living beings of a new

[(14)] A. Malraux, *Malraux face aux jeunes. Mai 68, avant, après*, ed. M. Crépu, Gallimard, 2016. On Berthelot's prospective vision, see his speech to the Chambre syndicale des produits chimiques on 5 April 1894, entitled "En l'an 2000" (repr. in *Science et morale*, Calmann-Lévy, 1897, p. 508-515).

[(15)] Cf. P. Spinicci, *Ithaque, enfin. Essais sur l'Odyssée et la philosophie de l'imagination*, French transl. J. Benoist, Vrin, 2018.

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kind, parts of nature one had not imagined. Odysseus' voyage, full of vicissitudes, could have unfolded differently; it is not a revelation of *the* truth. What Odysseus saw and lived through were not sublimated abstractions, but diverse things and different ways of living.

Moreover, keeping to the allegory, we can go further than Plato does with his plea for sublimation through ascent to the purified world of Ideas. Suppose that the prisoners in the cave had the culture of the "Renaissance men" and had written on the wall the cubic equation $x^3 + ax = b$. It may be that one of them (anticipating Scipione del Ferro) showed that the formula

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}}$$

was a solution in the case where the quantity $\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3$ is positive. One might then imagine that some of his companions pointed out that the equation can have three roots in the case where $\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3$ is negative, and that there is no formula in this case. This is where the excursion of the philosopher-prisoner comes in: he scales the back wall of the cave and emerges onto the sunlit path. What does he see there? He meets Tartaglia, Cardano, Ferrari, and above all Bombelli, who explain to him that in order to have formulas in this case, one must bring in the complex numbers, and that without these new symbols (which Bombelli designates by the vocables "più", "meno", "più di meno" and "meno di meno", with their products and their linear combinations), without this language it is impossible to have formulas for the three solutions.

Another example is provided by the physicist Niels Bohr, whose philosophy is an epistemology drawn from quantum mechanics. It is well known that Planck's principle — that energy at the atomic level is exchanged discretely in *quanta of energy* — is corroborated by the analysis of the absorption and emission frequencies of atoms according to the series of spectral lines (the Balmer series, Rydberg-Ritz, etc.), and that Bohr, representative of the Copenhagen School, was led to explain that a vision of the atom in which the electrons occupy quantised energy levels is compatible with the wave nature of light, which, for its part, is evident experimentally through interference, and theoretically through Maxwell's equations. Quantum mechanics presents several *dualities* that have the logical form of incompatibilities: between position and momentum with Heisenberg's uncertainty relations, through the diffraction of

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electrons by crystals (wave-particle duality), etc. The simplest is the dual wave-like and corpuscular nature of light, illustrated by interference and by the photoelectric effect, in which photons expel electrons from the metallic target. This *complementarity* occupies, in quantum theory, the place of a *paradigm* in Thomas Kuhn's sense. In these debates, Bohr's argument is fundamental:

> *Indeed, the spatial continuity of our picture of light propagation and the atomicity of the light effects are complementary aspects in the sense that they account for equally important features of the light phenomena which can never be brought into direct contradiction with one another, since their closer analysis in mechanical terms demand mutually exclusive experimental arrangements.*[16]

And he insists on this point in his discussions with Einstein:

> *However far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms. The argument is simply that by the word "experiment" we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangement and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics.*[17]

One sees that quantum mechanics occupies a very distinctive epistemological position. It typically consists of an excursion that starts from an experimental situation describable in classical physics, treats it with new concepts, and returns to a classical description. The corpus of quantum theory appears as a linguistic region that is called upon only temporarily and which nevertheless provides the best way of predicting what, in the end, happens in classical terms.[18]

[16] N. Bohr, "Light and Life" (1932), in *Atomic Physics and Human Knowledge*, New York, John Wiley & Sons, 1958, p. 5; French ed. used by the author: *Physique atomique et connaissance humaine*, transl. E. Bauer and R. Omnès, Gauthier-Villars, 1972, p. 4.

[17] N. Bohr, "Discussion with Einstein on Epistemological Problems in Atomic Physics" (1949), *ibid.*, p. 39; French ed., p. 58-59.

[18] Hermann Weyl wrote in 1931: "mathematics and physics make the world appear more and more as an open one, as a world not closed but pointing beyond

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In his popular philosophical essays on quantum mechanics, published from 1932 to 1957, Bohr on several occasions suggests parallels with other epistemological questions: a) with living beings in general, where the dependence of the living being on its context is the analogue of the link between the particles and the measuring apparatus; b) with biology, where the phenotype is the classical language to which the genotypic detour can lead. He presents these analogies with great caution, for it is in quantum mechanics only that the passage from one term of a contradiction to the other (complementarity) is achieved through a new auxiliary language bearing consequences for the results of measurements realisable at our scale. It is with good reason that he sees in it a rupture with the principle of classical causality:

> *The viewpoint of "complementarity" does, indeed, in no way mean an arbitrary renunciation as regards the analysis of atomic phenomena, but is on the contrary the expression of a rational synthesis of the wealth of experience in this field, which exceeds the limits to which the application of the concept of causality is naturally confined.*[19]

If only by virtue of the mathematical objects its language employs, quantum mechanics appears as *an extension* of the ordinary world. And the acceptance of such a typically heteropoietic procedure surpasses classical rationality.

This leads us to a third example, given by the approach to morphogenesis of René Thom, whose historical roots go back to Greek antiquity and Platonic philosophy.

Founded on his mathematical work on the classification of the singularities of differentiable manifolds, René Thom's approach broadens the scientific stakes through a language of forms, for the perception of nature and particularly in biology, for embryogenesis.[20] He has been reproached for having deviated from the

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itself. Or, as Franz Werfel expresses it in pregnant wording in one of his poems, *Diese Welt ist nicht die Welt allein*" (*The Open World*, Yale lectures of 1931, published 1932; preface, repr. in *Mind and Nature*, *op. cit.*, p. 34).

[19] N. Bohr, "Biology and Atomic Physics" (1937), in *Atomic Physics and Human Knowledge*, *op. cit.*, p. 19; French ed., p. 26-27.

[20] R. Thom, *Stabilité structurelle et morphogenèse*, InterÉditions, 1977; *Prédire n'est pas expliquer*, Eshel, 1991. See the fine synthesis by J. Petitot, "Les premiers textes de René Thom sur la morphogenèse et la linguistique : 1966-1970", 2015, HAL: hal-01265180v2.

empirico-deductive approach.[21] Indeed, the production of knowledge that he puts forward is an interpretative contribution and departs in this respect from strictly nomological science.[22] In this art of projecting parameterised kinematic models, the central paradigm is the reading of reality as issuing from a vaster, regular entity that is deformed by projection according to certain types. This master paradigm has its roots in Plato, Menaechmus, and above all Apollonius of Perga in connection with the conics.

Thom advocates an *explanatory* approach as opposed to purely quantitative modelling. He maintains that what he offers makes things intelligible, whereas assays and statistics are findings that still await interpretation. Can the *descriptive* have causal value in its own right? It is incontestable that, on the question of understanding, physicists do not seem troubled by certain aporias. Thom finds the quantum transition unintelligible — an electron jumping from one level to another while affecting the whole of space — and he protests:

> *Quantum mechanics is incontestably the intellectual scandal of the century! The fact is that science has renounced the intelligibility of the world; it has really renounced it! It is something that imposes itself and that is not intelligible.*[23]

For the moment, the general tenor of the articles published in biology is to rely on the properties of substances that are the chemical agents responsible for the circumstances under study. Causality is respected only relative to a framework which, as such, itself remains to be explained. The controversy between Thom and the biologists was, and still is, all the more lively in that the scientific methods that oppose each other present, in fact, a certain similarity:

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[21] Cf. R. S. Zahler and H. J. Sussmann, "Claims and Accomplishments of Applied Catastrophe Theory", *Nature*, vol. 269, 1977, p. 759-763; as well as J.-M. Lévy-Leblond, "Des mathématiques catastrophiques", *Critique*, vol. 33, 1977, p. 439.

[22] Cf. N. Bouleau, *Penser l'éventuel*, Quæ, 2017.

[23] *Prédire n'est pas expliquer*, *op. cit.*, p. 86.

the descriptive field attempts to find causal justification by mobilising idealities (Thom) or enzymes (biologists) whose legitimacy will have to be consolidated later.[24]

§ 5. — The new ancillary role of mathematics.

According to a view put forward by the neo-positivists, mathematics is a purely conventional system. Rudolf Carnap pointed out that mathematical geometries are invented, whereas physical geometry is measured. This is the old conception of mathematics as a "*servant discipline*": it serves to perfect the symbolic representations used by the natural sciences (in the sense of *Naturwissenschaften*).

But now biology, which studies living nature, does so on the basis of molecular representations in which combinatorics has a fundamental role and where, therefore, mathematics, as a formal system, has the status of a simplified nature.

The correspondence is as follows: the large molecules of living organisms — proteins (composed of amino acids), RNA, DNA — are thought of as mathematical *statements*. Let us recall that in first-order logic, a statement is a closed formula, that is to say, a formula without free variables; relative to a given interpretation, it is capable of being true or false. With the molecules *whose synthesis has succeeded* — whether by nature or in the laboratory — we associate the *proved* statements, in other words the *theorems*. *Chemical synthesis* is thus put in correspondence with *proof* in mathematics.

On the mathematical side, we can take the theory in which mathematics is most often practised today, namely set theory as axiomatised by Zermelo and Fraenkel (denoted ZF). But the simplest course is to take arithmetic as axiomatised by Giuseppe Peano with nine axioms defining the natural numbers, from which addition and multiplication can then be defined.[25]

[24] Another example of a heteropoietic detour might be the use of the symbolism of the mathemes by Jacques Lacan, as he presents them: "We haven't the slightest idea what they mean, but they are transmitted. Nevertheless, they are not transmitted without the help of language, and that's what makes the whole thing shaky" (Séminaire XX, *Encore*, p. 100; English transl. B. Fink, Norton, 1998, p. 110; quoted by N. Charraud, *Mathématiques lacaniennes à l'heure de Grothendieck*, Spartacus-IDH, forthcoming). And if we follow Claude Lévi-Strauss, shamanism can be understood in the same vein, as a heteropoietic journey ("L'efficacité symbolique", *Revue de l'histoire des religions*, vol. 135, no. 1, 1949, p. 5-27).

[25] Cf. N. Bouleau, *Introduction à la philosophie des sciences*, Spartacus-IDH, 2017, chap. II, § 2.

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If it is consistent, Peano arithmetic is incomplete and undecidable; more generally, the same holds of any consistent theory that is effectively axiomatised and sufficiently rich to formalise elementary arithmetic. The set of theorems is not given in advance; we augment it with each new proof. And if we are given a statement a priori, there is no computable bound on the length of the proofs one must try in order to prove it. Mathematics produces the *simple* from the complex; this is a fundamental trait of combinatorics.

This phenomenon appears only above a certain level of complexity. There is a threshold of axiomatic richness for incompleteness to appear. Of course mathematics follows the laws of logic, and indeed the set of these laws — first-order logic, traditionally called the predicate calculus — is complete[(26)]: every formula valid in all models is provable. Its general problem of validity is nonetheless undecidable: no algorithm decides, for every formula, whether it is valid. A consistent theory, effectively axiomatised and sufficiently rich to formalise arithmetic, is, moreover, incomplete: some of its statements are neither provable nor refutable. In simple language, from this level on, there is no general method for finding proofs. Mathematicians do not look for laws to produce theorems; they treat each conjecture one by one.

As regards the combinatorics of molecular biology, we know from more recent work by Y. Matiyasevich (1967) in particular that this phenomenology of incompleteness and undecidability is present, for the operations of cutting and splicing of DNA that occur there are amply sufficient to make it appear.[(27)]

Let us pursue this correspondence. On nature's side, there is production. Molecules are synthesised and some are new, owing to the genetic innovations in the course of evolution. On the side of arithmetic, it is the mathematicians who play this role; it is they who "synthesise" theorems. Yet nature does not start over from the initial axioms, those of three or four billion years ago; it starts from the living beings of today, which may moreover carry traces of very ancient things. But fundamentally, mathematicians do the same thing in their research work; they do not start each time from

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[(26)] Less famous than the incompleteness theorem, this result is also due to Kurt Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", *Monatshefte für Mathematik und Physik*, vol. 37, 1930, p. 349-360.

[(27)] Cf. *Ce que Nature sait*, *op. cit.*, p. 156-157 on Thue systems. See also W. D. Smith, "Computational Complexity of Synthetic Chemistry — Basic Facts", NEC Research Institute, working paper, 18 April 1997, 8 p.

the axioms, they use the theorems already proved (the fixed-point theorem, convergence criteria for series, etc.). Except for a total overhaul of mathematics such as the undertaking of the Bourbaki group, which remains exceptional.

There is nevertheless a difference here. The mathematics of the past is present in the mathematics of today. *We have not forgotten it*. We may have lost certain proofs when the library of Alexandria burned, but the results preserved, or cited by other authors, we know how to prove in other ways. It is different in nature.

The detours that nature took to obtain the successive living beings since the primordial soup are for the most part irretrievably lost. Palaeontology and genomic phylogeny give us only very fragmentary information about them, for the failures, the disappearances of species, the local or large-scale extinctions, have erased many successes that may have contributed to the syntheses useful to the survivors. The "proofs" carried out by nature to obtain the living beings of today are known to us only in fragments.

In any case, the syntheses successfully achieved by nature were carried out in a context that was itself evolving, whose details we shall never know. In mathematics it is quite different in this respect. The context of mathematical theories, the environment in which the proofs are immersed, is logic, and it is not perturbed by the appearance of new theorems. By contrast, in nature the relations to the context are crucial, not only at the level of living beings among themselves — the Darwinian level, so to speak — but also at the level of the framework within which the synthesis of proteins takes place, then cell differentiation occurs, etc.

I must insist on this point so as not to give the impression that I attribute to combinatorics an exclusive claim to explain the living world. That would be grotesque. The assertion that "the living being develops from DNA in a matrix and grows until it is the matrix of a new combinatorial process", which commands a broad consensus, obviously conceals the immense complexity of the major vital phenomenon of cell differentiation and of the role of epigenetic processes.

We see, then, that our isomorphism is a considerable simplification. We have left out an enormous number of factors — energy, bond strengths, reaction rates, catalysts, temperature, humidity, etc. — and, in speaking of a molecule whose synthesis has succeeded, we have not specified the orders of magnitude of the lifetimes involved.

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This simplified version of combinatorics nevertheless possesses a level of complexity sufficient to make apparent the limitations of our knowledge of living nature. We can think of this simplification as a first understanding. Just as an ideal gas follows thermodynamics with simple behavioural laws, our representation by mathematics has the advantage, by showing that the discovery of theorems is not something that can be done by algorithm, of opening up reflection beyond classical reductionism, without appealing to any mysticism or élan vital.

However, the example of an ideal gas is not completely adequate, for it might suggest that our model is an *approximation*, as is very often the case in engineering. Yet *in combinatorics it is not a matter of approximation*.

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§ 6. — The creative function and cognitive limit of biochemical heteropoiesis.

The detours of mathematical proofs, and the excursions of biochemists to carry out syntheses, mean that heteropoiesis presents itself from the cognitive point of view in a positive or a negative light — at times constructive and creative, at times on the contrary imposing limits on the knowable — depending on the data available and the freedom one has to use them.

If the conceivable detours are *ad libitum*, the creative function of heteropoiesis is at work when the aim — of the mathematician who proves or of the biochemist who synthesises — is the novelty of the result and of the path taken to obtain it. The empiricist then proceeds by trying out various chains with varied links, for there is no principle of purity of methods at this level of complexity. Chemists have long been aware of this. Thus Roald Hoffmann, in a fine article shortly after his Nobel Prize,[(28)] explains that often, in order to synthesise certain compounds, chemists are obliged to pass through much more complicated compounds. He emphasises that the discovery of these paths is a quasi-artistic activity that rests on the talent of the chemist and not on systematic methods. He takes as his example the synthesis of "cubane" C_8H_8, a molecule in the shape of a cube with a carbon and a hydrogen at each vertex.[(29)]

[(28)] R. Hoffmann, "In Praise of Synthesis", *American Scientist*, vol. 79, no. 1, 1991, p. 11-14.

[(29)] Cf. *Ce que Nature sait*, *op. cit.*, chapter II.

This gives synthetic biology and biochemistry a distinctive turn that calls into question classical epistemology founded on a nomological conception of knowledge. In a long three-part study, Hoffmann and Jean-Paul Malrieu discuss algorithmic methods, the relevance of simulation-based approaches, and attempt to circumscribe what belongs properly to human activity in chemical and biological synthesis.[30]

Very different is the situation in which evolution places us when we consider the living world as it currently appears to us and reflect on our possibilities and responsibilities, while most of the paths of synthesis that nature used to construct it have been irretrievably *erased*. A cognitive limit is marked by the absence of an *explicit* cause. There is a "that's the way it is" that imposes itself on us, a *being-there* that we observe and whose chemical origin we know, but which leaves us incapable of explicitly describing the chain of combinatorial causes. This is something other than Heidegger's *Dasein*; it would come closer to Hermann Weyl's *Dasein*. For we should have to say, *for nature as a whole*, that in its being its own being is at stake: the changes that nature effects are a function of what nature is, of what it has been, and of what it can accept, without our being able to make this fully explicit.

It is now established that the world is essentially chemical, but how is it that *this law* does not immediately open up to us the construction of a nature of our own choosing, techno-industrial, "which would supplant the old one"? Scientists certainly lack neither desires nor impulses.[31] a) We do not possess the proofs of all the molecules of the living world of today. b) What is not part of the observable biological present is also chemical, and it remains unknown to us until it has revealed itself through its effects, arising from biochemical relations with what already exists.

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[31] "The damage is done, in the sense that, supported by the development of scientific knowledge (and ideology), the new techno-industrial nature has supplanted the old one", letter from Ph. Courrège to A. Grothendieck of 1 March 1972.

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