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Book I — Theory of Sets

Introduction

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It is in the nature of a mathematical demonstration to demand the reader's complete and unreserved assent. This is a requirement whose absolute and total nature is perhaps not sufficiently emphasised in teaching; the difficulty encountered by many otherwise excellent minds in understanding mathematical reasoning may be largely due to the fact that they have not fully understood that each new theorem, if they accept it at the time, must also be accepted with all its consequences, even distant ones. Non-mathematical reasoning, it seems, exhausts its powers after a certain number of steps; as soon as a chain of deductions becomes lengthy, the mind quickly feels the need to check its operations by some direct experiment, or at least by another convergent argument. Thus, the philosopher believes that the truth of a given conclusion, say the existence of God, is more firmly established if he has several different proofs for it. Mathematicians, on the other hand, consider a theorem to be true as soon as they have a single proof; any other proofs they may subsequently provide may be simpler or more elegant, but they add nothing to the truth of the theorem, which is indisputable once it has been established.

It is undoubtedly easy for the mathematician to mock the philosopher for the lack of certainty in his conclusions and for not yet having definitively solved a single one of the fundamental problems that have continued to arise for centuries. It is also easy for philosophers to rebel against an attitude they consider dogmatic and contrary to what they see as their duty to never accept anything definitively and to subject all opinions to perpetual critical examination.

But mockery and mutual hostility do not help us to understand better. They are all the more out of season since mathematicians have never been more committed than today to criticising the very foundations of their thinking, and since, on the other hand, the existentialist current in modern philosophy tends to emphasise the need for an initial choice and total commitment, whereby the subject assumes absolute responsibility and refrains from critical backward glances.

It will be all the more timely to attempt to shed light on this paradox given that this very generation of mathematicians who, even more than their predecessors, emphasise the ideal of rigour in demonstrations and, by the same token, the requirement for absolute truth in theorems, is also the one who engages in the most critical examination—and, in many respects, the most destructive—of the very foundations of mathematics. The mathematician is both uncompromising and non-dogmatic. We are led to wonder whether the surprising coexistence of these two attributes can be explained by a difference between the meanings that mathematicians and ordinary people attach to the word 'truth'.

A judgement is true, it is generally said, when it conforms to the nature of things, when it reflects reality. Is it true that the sum of the angles of a triangle is equal to two right angles? The empiricist will answer: "Yes; if you doubt it, try it out with a number of triangles that you have drawn". The Platonist will answer: "Yes, because the property of having the sum of its angles equal to two right angles is inherent in the nature of the idea of a triangle." But the mathematician will say: "Yes, here is the proof." So far, there is no radical divergence; the empiricist, if he wants to remain so at all costs, will say that experience has shown that judgements deduced by the usual logical rules of true judgements (in accordance with experience) are still true; the rationalist will admit that our intellect participates in the world of ideas and, as a result, that we are able to link together intelligible truths. But here is another theorem: the set of points on a straight line is uncountable. There is no question of experimental verification here; the very statement of the theorem would imply an operation that is impossible in principle, that of counting the elements of an infinite set. And it is a fact that several mathematicians, including some of the greatest, have argued that in such statements, thought is carried away by a verbal game that falls short of the limits of its effectiveness, losing touch with reality. But these criticisms have not prevented the vast majority of mathematicians from using the admirable tools that Cantor has made available to them. Is it, the pragmatist will ask, precisely because these tools are useful that the mathematician uses them? But a method is only useful to the mathematician if it allows him to prove *true* theorems; if it were only a matter of obtaining a large number of theorems, or very beautiful ones, the slightest error would be much more useful than any theory: it would allow *all* theorems to be proven. Should we therefore think that accepting the above theorem as valid implies a conversion of the mathematician to the doctrine of the real existence of an intelligible world that actually contains infinite sets of different powers? It is very remarkable that this is not the case at all. A large number of mathematicians believe that the actual existence of infinite sets belongs to a realm of speculation that they dismissively refer to as "metaphysical"; by this they mean a sphere of thought in which judgements are devoid of certainty and, according to some, even of meaning. Imagine a geneticist who asserts as absolutely true that the human cell contains 48 chromosomes, but who rejects as "metaphysical" and devoid of certainty or meaning the assertion that cells and chromosomes actually exist! How, then, can many mathematicians remain virtually indifferent to the question of the real existence of the objects referred to, or seemingly referred to, in the theorems they prove with such rigorous care?

To try to understand this, let us imagine a mathematician who is as firmly convinced of the real existence of sets as the naturalist is of the existence of living beings. It would seem that, for this mathematician, definitive confirmation of the truth of a theorem could only result from a direct intellectual vision of the objects referred to in the statement. Just as the naturalist strives to improve the experimental techniques that enable him to *see* the phenomena of life unfolding in the field of his microscope, so one of the main concerns of the fictional mathematician we are trying to imagine would be to develop a kind of yoga that would enable him to gain immediate knowledge of the properties of mathematical objects. No doubt he would still resort to demonstrations, insofar as the imperfection of the human brain would prevent him from arriving at a direct intuition of the truth; but he would be constantly concerned with checking the results he obtained in this way through experimentation (which for him would be a kind of internal experience). If the conception whose consequences we are trying to imagine reflected the true nature of mathematics, teaching materials would include (as in physics) a description of experimental methods: regulation of the respiratory system, complete immobility, successive degrees of meditation, all the stages of access to the mystical life would be described. Our universities would have research laboratories where mathematics students would learn to practise inner concentration, just as chemistry students learn to work with glass and biologists learn to use a microscope. An original thesis containing only demonstrations of new theorems would have the value of what is called in physics a "theory", that is, a means of predicting experimental results; this theory would only be truly accepted as true on the day when an ingenious form of yoga made it possible to perceive its conclusions *directly*.

Such may be the mathematics of the future. The example of the prodigious development of experimental methods over the past five centuries and the fact that Eastern methods of inner concentration are still poorly understood and have never been applied in the field of scientific research caution us against an attitude that would deny *a priori* the possibility of such a development. Perhaps our successors will view today's mathematics with the same condescending disdain that we view the deductive physics experiments of the Middle Ages. But we sense that, if this is the case, it is because a veritable revolution will have overturned our conception of mathematics. The fact that modern mathematics bears practically no resemblance to the image we have tried to form forces us to conclude that the belief that some mathematicians today profess in the real existence of mathematical beings is a wholly abstract belief; an opinion, not a faith; it plays almost no role in their activity as mathematicians.

We are thus led to think that the definition of the truth of a judgement as conformity to a real order of things is inoperative in the case of mathematical assertions. The decisive and, in the final analysis, only argument in favour of the truth of a theorem is to provide a proof of it, that is, to show that its statement is the last in a sequence of logically linked sentences. A verbal game, the intuitionists objected earlier; a verbal game indeed, many modern mathematicians would say, just as the Whigs came to take pride in the name their opponents maliciously called them. And it must be agreed that what makes a proof powerful is the very perfection with which it follows the rules of the verbal game, as codified by logic. The use of intuition has a bad reputation in mathematics; far from constituting perfect proof of truth, as in the fictional science we imagined above, it is, on the contrary, considered a serious mistake that can invalidate the scope of a proof (if it is not accompanied by logical reasoning that guarantees its validity). Here we touch once again on the source of the difficulty mentioned above, which many people experience in following mathematical reasoning: they have not been told that *understanding* a proof and *accepting it* are two very different things. Understanding an elementary geometry proof means relating its successive steps to the spatial intuition of geometric objects; it means, in a way, seeing it unfold before our eyes. But accepting it means verifying, step by step, that each new assertion is indeed a logical consequence of the previous ones. However, mathematicians require us to reject certain "proofs" that we do understand (this is the case when the pseudo-proof in question contains certain assertions that can only be justified by intuition about the objects in question) – and even to accept proofs that we do not understand: which is why people who are completely incapable of "seeing in space" can nevertheless do three-dimensional geometry. The ambiguity is further perpetuated by the fact that making a proof understandable is the best (if not the only) way to convince the listener that the result is correct. And this remark does not apply only to elementary education; a mathematician who communicates his results orally to a colleague seeks to persuade his interlocutor by translating the terms of his speech into a language that appeals to the intuition of the person he is talking to. Nevertheless, there remains a tacit agreement between the two that nothing is done until a formal proof has been written. This proof will undoubtedly not reproduce all the necessary logical sequences in their entirety, but these omissions are only to avoid lengthy explanations, and the author's duty as a mathematician is to always be ready to provide any details upon request. It sometimes happens that the abundance of details contained in a written proof, or the clumsy manner in which they are written, detracts from its intelligibility to such an extent that one is occasionally forced to accept proofs that one does not understand. This causes deep intellectual discomfort, but the mathematician who has been forced to follow the steps of a proof without understanding its guiding ideas does not question the validity of the result, even if he or she bitterly attacks the inexperience of the writer.

It is easy to understand why the requirement for *absolute* certainty that is characteristic of mathematics leads to the prohibition of the use of intuition. This is because intuition is never perfect; if it were, it would be immediate and the most distant consequences would be apparent at first glance, without the need for demonstrations to arrive at them. The partial nature of our intuitions is of secondary importance in the natural sciences or philosophy, where subsequent experiments or intuitions can go back and gradually refine the results that an initial, perhaps crude, intuition may have led us to state. But this is not the case in mathematics, whose motto is "all or nothing". Mariotte's law is nonetheless a good physical law, even if it is not entirely correct; physicists will know how to make the necessary corrections whenever they use it, depending on the circumstances. But there is no question of making corrections to a previous theorem that is used in a demonstration. The slightest error would be multiplied beyond any controllable limit in the subsequent reasoning and would radically vitiate the often far-reaching conclusions drawn from the theorem in question.

We must now examine the conditions for the possibility of a "pure verbal game", a demonstration that we can be sure does not appeal to intuitions foreign to the very process of reasoning. Now, it is quite clear that the danger of a lack of rigour lies in the *meaning* that words have for our minds. It is through their meaning that they are still connected to an uncertain world, to an opaque and fluid existence of which our minds cannot gain adequate knowledge. The very force of the movement of the mind that sought to establish a certain science thus pushes it to break the ties that bind words to the things they represent, to empty them of their meaning, to make certain science a certainty about nothing. A pure verbal game is a game with words that mean nothing. Before explaining in more detail how mathematicians can conceive of operations on words deprived of meaning, let us pause for a moment to note that the movement we have just described, strange as it may seem at first, is not unique to mathematics; it even seems to be inherent in principle to the very act by which language constitutes itself as such. We refer here to the very profound analysis of the conditions of literature by an eminent contemporary critic, Mr Blanchot. The word, observes Mr. Blanchot, serves at least as much to distance us from the thing it signifies as to bring it to us: "in order for me to say: this woman, I must, in one way or another, strip her of her reality of flesh and bone, render her abstract and annihilate her"(2)—and again: "God created beings, but man had to annihilate them. It was then that they took on meaning for him...". The creation of words therefore already creates a gap between the word itself and the thing it signifies (the question of whether the thing itself is not constituted as a separate being by the way we look at it from the other side of the gap is another question that we will not address here). In fact, everyday language preserves a kind of communication between the word and the thing: "... the word still relates to it (to the existence of what it designates) through the non-existence that has become the essence of that thing. To name the cat is, if you like, to make it a non-cat, a cat that has ceased to exist as a living cat, but that does not make it a dog, or even a non-dog" - everyday language admits that once the non-existence of the cat has passed into the word, the cat itself is fully and certainly resurrected as its

(2) *Translator's note*: M. Blanchot, « La littérature et le droit à la mort », in *La part du feu*, Paris, Gallimard, 1949.

idea (its being) and as its meaning...". However, literature, says Mr. Blanchot, does not stop there, and neither does mathematics, as we have just seen. In both cases, language "is tempted... to want to reach negation itself and to make everything out of nothing. If we only talk about things by saying about them that by what they are nothing, then saying nothing is the only hope of saying everything about them."

But what remains of a word when its meaning is removed, if not its material structure? The fact that words are also things can be seen, from a certain point of view, as a failure of language to constitute itself as a set of meanings: the letter obscures the spirit, that is to say, attachment to words for their own sake always risks obscuring the message that words are meant to communicate. But if we renounce meanings from the outset, then the physical reality of words becomes, as M. Blanchot says, "my only chance". At this point, in fact, the paths of literature and mathematics diverge somewhat. For the poet, it is the words themselves that become essential, with all their individual characteristics: "Everything physical plays a role: rhythm, weight, mass, shape, and then the paper on which we write, the trace of ink, the book". For the mathematician, on the contrary, the focus is not on the individual qualities of words but on their articulation in sentences. Syntax remains possible after the act that has driven away meanings; just as one can still play with those assemblages of cut wood called "puzzles" by turning the pieces face down on the table so as not to see the fragments of images drawn on them. Mathematicians will, moreover, abandon words themselves and replace them with *signs*, which are more manageable; he is more or less forced to do so because the rules of syntax are far too imprecise for the use he intends to make of them, and also because ordinary syntax is not pure syntax, but also makes its prescriptions depend on the meaning of the sentences to be formed.

We must now go a little further in describing what mathematics entirely devoid of meaning might be. To understand this, we can use an analogy with elementary algebraic operations. To simplify matters, we will only consider the calculation of algebraic expressions involving addition or subtraction (without multiplication or division), and we will only consider this calculation insofar as it aims to establish algebraic identities (not to solve equations). We then find formulas, each of which asserts the equality of two algebraic expressions, written on either side of the $=$ sign.

It is generally taught that the letters in these expressions represent numbers, that the signs $+$ and $-$ are symbols for operations to be performed on these numbers, and that algebraic identities are formulas that express the equality of the values taken by two algebraic expressions for all possible numerical value systems of the letters appearing in them. But the technique of calculation consists precisely in disregarding all these meanings and operating directly on the algebraic expressions without worrying about whether they have numerical values. When we replace $-(a+b)$ with $(-a)+(-b)$, we do so automatically, without thinking about the theorem that states that the opposite of the sum of two numbers is the sum of the opposites of those two numbers. However, we could imagine codifying once and for all these automatic processes that long practice in calculation has implanted in our minds. We would then have a system of normative rules for algebraic calculation that would be entirely independent of the interpretation of algebraic expressions as symbols for operations to be performed on numbers. These rules would first teach us to recognise an algebraic expression (because a combination of signs such as $a{+}{+}$ is not one). We would therefore have statements such as the following: any letter is an algebraic expression; the sign 0 is an algebraic expression; if E and F are already written algebraic expressions, the combinations of signs $-(E)$ (i.e. the sign $-$, followed by the sign (, followed by the transcription of the expression E, followed by the sign)), $(E)+(F)$ and $(E)-(F)$ are algebraic expressions; it would also be specified that the parentheses surrounding E can be removed if E is a letter or the sign 0. It would be agreed to generally refer to all combinations of signs that are algebraic expressions by virtue of the above rules applied one or more times as algebraic expressions.

We would then have rules for writing identities, of which here are a few examples (the list is not exhaustive): if E and F are algebraic expressions, the combinations of signs $E = E$, $(E)+(F) = (F)+(E)$ are identities; if $E = F$ is an identity, then so are $F = E$ and $(E)-(F) = 0$; if the combinations of symbols $E = F$ and $F = G$ are identities, then so is $E = G$. We would then call algebraic identities all the combinations of symbols that could be obtained by repeated application of the previous rules.

That said, formal mathematics would have a structure very similar to that of the codified algebraic calculus we have just imagined. It would first include rules for forming sentences, specifying which combinations of signs will be called sentences, and rules of deduction, allowing us to write true sentences, or theorems. In a system based on set theory, we would have, for example (among others), the following formative rules:

- The combination of symbols obtained by writing two letters, one to the left and one to the right of the symbol $\in$, is a sentence (this is the formal transcription of the assertion that the object represented by the letter on the left belongs to the set represented by the letter on the right);
- if P and Q are sentences, the combinations of symbols $\sim (P)$ and $(P) \Rightarrow (Q)$ are sentences (the first is the formal transcription of the negation of the assertion of which P is the formal transcription; the second is the formal transcription of the assertion that the assertion transcribed by P cannot be true without the assertion transcribed by Q also being true);
- if P is a sentence, the same is true of the combination of symbols $(\forall x)(P)$, or any other combination derived from the previous one by replacing x with any other letter ($(\forall x)(P)$ is the formal transcription of the assertion that the assertion transcribed by P, which refers to the object x, is universally true, i.e. true for any object x).

The formal rules we have just cited do not constitute a complete system; we have given them only as examples. Furthermore, the word "letter" that appears in them must be qualified. The use of the 26 letters of the alphabet is not sufficient for formal mathematics; we therefore agree to call a letter any assembly of signs formed by a letter assigned any number of primes; thus x, y', z'', a''', ...

The rules of deduction allow certain sentences to be called theorems. Here are two examples:

a) the set of symbols obtained by writing the same letter on either side of the $=$ sign is a theorem;

b) if the sentences P and $(P) \Rightarrow (Q)$ are theorems, then Q is a theorem.

The two examples we have just cited present an obvious difference between them. Rule a) allows certain theorems to be written without any prior knowledge; it is said to constitute an axiom scheme. Rule b), on the other hand, is only applicable if we already know that certain sentences are theorems.

A mathematical text composed of sentences written in a specific order is called a proof if the rules of reasoning allow us to establish, step by step, that each sentence in the text is a theorem. The first sentence must then naturally be an axiom, i.e. it must be a theorem by virtue of a rule, such as a), whose application does not require knowledge of any prior theorem. On the other hand, the following sentences in the text may be theorems by virtue of a rule such as b) and a number of sentences that already appear in the proof and have therefore been recognised as theorems. The last sentence of a proof is called the conclusion of the proof.

It should also be noted that the statement of the rules of reasoning is not part of formal mathematics; the rules constitute a kind of instruction manual for a mathematical text. These instructions can be understood as applying to either the author or the reader: in the first case, they indicate the conditions for writing a correct text; in the second case, they provide a means of verifying that a given text is indeed correct. Formative rules are generally such that they allow one to recognise by simple inspection whether a given assembly of signs is a sentence. On the contrary, the rules of deduction do not allow us to decide by simple inspection whether a given sentence is a theorem; but they do allow us to recognise whether a given sequence of sentences is a proof. To make this verification easier, a commentary on the text (in ordinary language) can indicate for each sentence, by virtue of which rule of reasoning and which previous theorems this sentence is a theorem.

We have given only a very brief and partial outline of what formal mathematics might consist of, a "pure verbal game". We will now try to estimate the value of such a construction for the mathematician.

One remark must be made from the outset: such mathematics does not actually exist. Standard mathematical texts are, of course, written in everyday language, governed by the laws of ordinary syntax and grammar. Furthermore, even specialists in formal logic have never presented a mathematical text or even a system of rules (formative rules and rules of reasoning) that strictly conforms to the requirements of pure formalism. One of the reasons for this seemingly surprising deficiency is the extreme complexity that such a system of rules would entail. To remedy this complexity, abbreviations are introduced, which are new signs that are no longer devoid of meaning, as those in formal mathematics proper, but which represent other signs or combinations of signs; however, it is clear that a text that uses these abbreviations is no longer a purely formal text. Moreover, and above all, there is no pressing need today for mathematicians to undertake the considerable work involved in writing a book that strictly conforms to the canons of formal mathematics. Ordinary proofs are considered to be free from untimely recourse to external intuitions; as no serious objections have been raised in their regard, it is not considered necessary to replace them with formal proofs. Mathematicians are much more interested in finding intuitions to avoid than in ways of avoiding them.

Nevertheless, formal mathematics constitutes a kind of horizon for real mathematics. The possibility, conceived as always open, of formalising proofs plays a fairly significant psychological role by guaranteeing the existence of a kind of Maginot line, supposedly impregnable, to which it would always be possible to fall back in the event of serious dispute. But is this ideal guarantee philosophically justifiable? This is what we must now examine.

We have seen how the requirement for absolute certainty in proofs makes any recourse to intuition in the development of mathematics suspect. Real beings, by the very fact that they are real, are opaque beings that cannot be fully illuminated by the light of the mind; only consciousness is entirely transparent to itself. We therefore had to detach mathematical words or signs from their meaning and operate on them according to purely formal rules in order to ensure the possibility of rigorous reasoning. But in trying to be angels, have we not become beasts? For once words are deprived of their meaning, we have no hold on them except through their physical appearance, their material constitution. We have not eliminated concrete intuition at all; *we have merely replaced the intuition of what the words meant with that of the perceptible aspect of the symbols*. And, in fact, the rules of formal mathematics are addressed not to a pure mind, but to a mathematician who *sees* the symbols. Their application presupposes the ability of the eye to recognise the signs from one another, to recognise the identity of the same sign in different places on the page; the words "left" and "right" are used freely; we are asked to be able to distinguish the successive lines of a text, etc., etc. Has the mind not re-engaged with a world of real objects, and can we assert that signs constitute a privileged domain of the sensible world that is entirely transparent to the intellect? There is no such thing as a purely formal mathematical text; but what comes closest to it would undoubtedly be a long calculation, say in analytical geometry (we have already recognised above the analogy between the rules of deduction and those of elementary algebraic calculation). But does the mind feel irresistibly compelled to accept a result that is the outcome of a calculation of considerable length? We mentioned earlier the unease left by a proof that is accepted but not understood; we suggested that this unsatisfied desire to understand was a kind of luxury, a pleasure that the intellect offers itself, but that understanding the ideas in a proof adds nothing to its binding force. The value of certainty in a proof lies, we say, solely in the rigorous logical articulation of the assertions of which it is composed. But we have reduced the operations by which this logical correctness is verified to a series of material comparisons of written signs; should we not then revise our opinion, taking into account the possibility *of material errors*? Anyone who has ever done calculations has made mistakes, and most often these are very simple mistakes, such as transcribing one number instead of another that was misread. These errors are usually discovered and corrected after a certain amount of time, most often because they lead to unacceptable results. We then retrace the chain of calculations, check each step, and correct our mistakes. But doesn't this mean that, as in physics and philosophy, the mind goes back and corrects faulty intuitions in the light of later intuitions? The fact is that most of the errors that can be made in an analytical geometry calculation are revealed by the absurdity of the *geometric* consequences they entail, that is, by their contradiction with certain intuitions of a nature quite different from that of written signs. Does this fact not invite a radical attitude of mistrust towards demonstrations that claim to be purely formal and which, as a result, renounce any intuition other than that of mathematical symbols?

These remarks may seem discouraging; however, should we conclude that formalism was a false hope, and of no value to mathematicians? That would be like saying that analytical geometry is a negligible aspect of geometry: no mathematician would support such a paradox. It should be remembered here that the intention of formalism in its early days was not to eliminate all recourse to intuition, but only to show that it was possible to dispense with the consideration of infinite sets thought to have an objective existence. This limited objective can now be considered to have been achieved. Since the work of von Neumann, Bernays and Gödel, we have been in possession of a language for set theory in which it seems at least probable that the famous "paradoxes" cannot be formulated. We are able to prove the theorems of set theory and, as a result, apply them in other branches of mathematics without having to worry about whether the sets we are talking about exist and whether the axioms we start from accurately reflect their real properties. It is true that, despite this success, the current state of affairs cannot be considered satisfactory. The intuition of a set as a collection of all objects possessing a certain property remains, we believe, present in the mind of the mathematician, even when formal reasoning makes no mention of it. However, if we examine the axioms of set theory not formally, but from the point of view of this intuition, which is in fact always present, we find that they do not entirely fit with it. Whether it is Gödel's distinction between "classes" and "sets", or other similar artifices that can be used to avoid paradoxes, we always find in axiomatic systems that we are led to formulate certain restrictions whose artificial nature causes a kind of intellectual discomfort that is difficult to overcome. If the function of words is to drive out things and free the mind from the oppression of a presence that is too close, they must *also* restore things - "through the non-existence that has become the essence of that thing", as M. Blanchot puts it. But we sense that the axioms of set theory do not adequately restore its essence. Is this a definitive shortcoming, or can we hope that, following Gödel's opinion quoted by H. Weyl, "our logical optics is only slightly out of focus" and that "after some minor correction of it we shall see *sharp*, and then everybody will agree that we see *right*"? The future will undoubtedly decide.

But it is not only in set theory that the conception of the possibility of a purely formal demonstration gives mathematics a new creative vigour. The movement by which reasoning detaches itself from the intuition of a particular object not only has the effect of freeing the conclusion from the doubtful character that our insufficient knowledge of the object in question might have conferred on it; it also has the effect that the proof, now free from its initial object, can be applied to other objects. This is the foundation of the *axiomatic method*, which is known to play an immense role in modern mathematics. It is based on the recognition of similarities in logical structure that can be found in theories that are apparently very different. Once these similarities have been recognised, several analogous proofs relating to different objects can be replaced by a single proof relating to an indeterminate object which is postulated to have just the right properties for the proof to be carried out. This approach is, of course, only legitimate because the proof obtained is formal, i.e. valid regardless of the nature of the object to which it applies.

The axiomatic method is therefore primarily a method of economy of thought, in that it allows several different lines of reasoning to be condensed into one. M.R. Queneau, in his "Exercises in Style"(3), set out to give multiple accounts of the same incident; the mathematician, on the other hand, learns to give the same account of multiple incidents! But the axiomatic method is also a method of discovery and progress. This is because intelligence, freed from the grip of objects that constrained it too tightly, paralysing its activity in a world too real to be translucent, regains an agility that allows it to leap towards new conquests. Just as the Aztec gods prolonged their existence only by drinking the blood of the victims sacrificed to them, so too does the intellect require a constantly renewed sacrifice of reality in order to continue its forward march.

We said earlier that one of the main reasons why there is no such thing as pure formal mathematical literature is the extreme complexity that such mathematics would have to present. Furthermore, we added that even if such mathematics did exist, its conclusions would not even have the character of absolute certainty that we would be entitled to expect, due to the impossibility of being completely protected against the risk of calculation errors. However, against the danger of suffocation posed by a world too complex to be immediately understood, the mind has a defence, always the same, which is to *name* the things that constrict it too closely; that is, to keep them at a distance, to substitute their essence for their existence, and, in the void thus created, to regain its freedom of movement. And since, in the present circumstance, it is words, these word-things deprived of their meaning, that refuse to let it penetrate them with its gaze, the possibility remains of naming the words and their combinations. This is how what is called metamathematics is born.

All works on modern formal logic introduce, at the beginning of the exposition, abbreviations that are not meaningless symbols, as in formal mathematics, but each of which represents a combination of meaningless symbols. The "sentences" in which such abbreviations appear are neither sentences of formal mathematics (since

(3) *Translator's note*: R. Queneau, *Exercices de style*, Gallimard, Paris, 1947.

they refer at least partially to meanings, rather than merely existing on paper in their material presence as entities in themselves) nor sentences of everyday language (since they also contain meaningless symbols, whereas all the words in a sentence in everyday language have meaning). In order to avoid the ambiguity of such "sentences", it would undoubtedly be preferable to restore to them entirely the value of non-formal *nouns* that would designate formal sentences. Once this step has been taken, we are fully in the realm of metamathematics, which can be defined as the study of sentences (and more generally of sequences of sentences and in particular of proofs) of formal mathematics taken as the object of descriptive thought.

What makes an undertaking such as founding metamathematics quite unique is that the formal mathematics it proposes to study exists, as we have seen, only as an ideal possibility. Metamathematics is therefore a descriptive science of a purely imaginary object. Its purpose is, among other things, to study the question of which theorems could be proven in formal mathematics (if it existed) and, in particular, to determine whether it is true that formal mathematics is non-contradictory (i.e., whether there is no theorem whose negation is also a theorem). What means does metamathematics have at its disposal to address its subject? It must be able to discuss all possible combinations of formal symbols, which are naturally infinite in number. It is therefore natural to consider using the integers of ordinary arithmetic for this purpose; the latter will then function not as formal mathematics, but as applied mathematics (the integers being not meaningless symbols, but retaining their counting power). Having thus designated each possible proof by a number, we can reason about these proofs in terms of elementary arithmetic. This is the method followed by Gödel; it led him to certain results of very general significance, which we will now discuss briefly. The question whether formal mathematics is not contradictory remains open. But, assuming that it is not, we know that it will forever be impossible for it to establish for itself that it is not. Let us explain more precisely what this means. We saw earlier how elementary arithmetic could be used to reason in metamathematics. But elementary arithmetic can also be considered part of formal mathematics. It is therefore possible to translate metamathematical assertions into formal statements and metamathematical reasoning into formal proofs (provided that they do not use any means other than those available to mathematics). That said, Gödel's theorem states that the formal translation of the (non-formal) statement "mathematics is not contradictory" cannot be the conclusion of any formal proof (unless mathematics is contradictory). Furthermore, Gödel also showed that, still assuming mathematics to be non-contradictory, it is possible to write a formal sentence P such that neither P nor its negation is a theorem; we can therefore state problems that we know cannot be solved. More precisely, we can state (in formal terms) a property $P(x)$ of an arbitrary integer x such that, for any given integer a, the sentence $P(a)$ (which asserts that this integer has the property in question) is a theorem, whereas the formal sentence that expresses the assertion that "every integer x has the property $P(x)$" is not a theorem; the negation of this sentence is naturally not a theorem either.

This last result is of considerable interest for the following reason. We said above that formal mathematics, by dispensing with the intuition of mathematical beings, allows us to prove many more theorems than those who want all statements to refer to real beings accessible to intuition are willing to admit. On the other hand, Gödel's result shows that certain theorems can be true from an intuitive point of view without being true in formal mathematics. For if we do not disregard the meaning of the words "for every integer x", it is clear that we will accept as true that "every integer x has property $P(x)$" if this judgement of fact is consistent with reality, i.e. if, in fact, all integers have the property in question. However, Gödel's results cited above apply not only to a given system of formal mathematics, but to any system rich enough to formulate elementary arithmetic. This means that any attempt to replace deductive reasoning with a purely verbal game can only aim at a partial substitution: there are non-formal methods of demonstration based on intuitions that no one, it seems, can dispute, which are more powerful than any possible formal reasoning.

We should not conclude this examination of the results of metamathematics without citing a theorem that is more positive in scope than the results stated above. It was metamathematics that provided the first information on a purely mathematical question in set theory, namely the continuum hypothesis. We will not formulate this hypothesis here; suffice it to say that it is an assertion whose truth (i.e., that it is a theorem) is not yet known. However, Gödel showed that if set theory is not contradictory, we can assert that the continuum hypothesis is not false, i.e. that its negation is not a theorem of the theory. This is a considerable advance in the study of a question that has intrigued specialists in set theory for more than half a century.

We have previously been led to reject mathematics' claim to absolute certainty. Even if it were purely formal, it could not completely protect itself against the dangers of material errors. And, as far as mathematics currently exists, Gödel's results clearly show that even translating it into formal language is not an operation that can be carried out without serious controversy; we have seen that interpreting assertions about *all* integers, for example, in formal symbols cannot constitute an adequate translation. However, we should not conclude from this that the possibility of formalisation that remains on the horizon of current mathematics is of no practical value in clarifying differences of opinion or limited controversies. It is always possible to carry out *partial* formalisation, thanks to the use of appropriate symbols and operational rules relating to these symbols. If doubts arise about the scope of a particular definition or theorem, and in particular about the meaning of the words used, it is always possible to clarify what is being discussed by indicating, at least in part, the nature of the formal operations that are to be performed. This approach may sometimes be preferable to explaining the meaning attributed to the disputed word by referring to intuitions that are not necessarily the same in different minds. If there is a real divergence, there is a chance that the method of partial formalisation will make it possible to pinpoint the debate and find where definitions need to be clarified in order to restore agreement among mathematicians on the scope of the theorems stated.

But there is more. It seems that the concept of a possible underlying formal mathematics, a concept that is almost always present in the background of contemporary mathematical thought, has the effect of profoundly altering the very intuitive content of the theorems. The notion of topological space was first abstracted from the general idea of continuity; it is images relating to the structure of sensible space that first haunt the mathematician's mind when he begins to reason in topology. But these images of a malleable space pierced with holes disappear, we believe, fairly quickly from consciousness, to be replaced by intuitions of possible formal operations: a topological space soon ceases to be a rubber ball in the mind and becomes a space about which *we can talk* about continuity, convergence, connectedness or closed sets. The coexistence of two kinds of intuition is quite noticeable in certain cases where two equivalent definitions are possible, such as that of differentiable manifolds. For the chosist[4] intuition, a differentiable manifold is a space composed of pieces of Euclidean space connected according to a certain law, but for the formal intuition, it is a space about which *we can speak* of differentiable functions. Both points of view lead to formal definitions, which are moreover equivalent, of the notion of a differentiable manifold, but it is clear that the meaning of what is defined is not the same in both cases. Finally, there are cases where the content of thought relates almost exclusively to the formal aspect of the notion under consideration. Thus, a mathematician, when considering the content of his idea of *isomorphic* mathematical beings, will, we believe, think much less of a complete similarity between two objects as things than of the following: any theorem relating to one of these objects *can be translated* into a theorem relating to the other. An even more striking example is that of a concept such as the "exact sequence" of groups and homomorphisms; in this case, the chosist intuition of objects that would be the groups of the sequence disappears almost entirely in favour of an intuition that relates only to a certain number of methods of proof that can be used in various circumstances.

The importance of formal intuitions (i.e., those related to formalism) in mathematical thinking is further revealed in the examination of figures that sometimes aid in understanding the text. The figures in a work on elementary geometry represent the very objects—lines, circles, triangles, etc.—discussed in the theorems: these are chosist figures. But there is also another type of figure whose purpose is to represent not the objects themselves but a certain number of formal connections. Thus, the theory of commutative rings and fields often uses diagrams that represent the inclusion or isomorphism relations between the objects being discussed. However, these figures do not represent, as one might think, a sub-field by a part of a space whose totality would represent the field in which it is contained; on the contrary, the sub-field and the field in which it is contained are represented by letters that are joined to each other by an arrow-shaped sign, the same as the one that represents a homomorphism. The intuition that such a figure appeals to is one that empties the idea of inclusion of its primary

[4] *Translator's note*: Bourbaki opposes « *choses* » (things) to forms and correspondingly a « *chosiste* » intuition to formal intuition. We translate the French adjective « *chosiste* » by "chosist" that we prefer to "thingist" ("thingism" being used to translate « *chosisme* »).

meaning (relationship between container and content) in order to retain only certain formal properties that bring it closer to that of homomorphism.

Recognition of the growing importance of this intuition of forms of reasoning in current mathematics could prompt us to attempt a completely new interpretation of our science, according to which it would in fact be identical to metamathematics. We could then put forward the hypothesis that the theorems found in mathematical works are not, as the intuitionists understood them, statements that speak of given real objects, or, as the formalists understood them, statements devoid of meaning and substitutes for formal sentences that are too long to write, but rather statements that, without *being* formal sentences, at least *refer* to formal sentences. One would indeed be asserting something in a theorem; but what one would be asserting would not be a property of the real objects one seems to be talking about, but rather the fact that (formal) mathematics contains a (formal) proof of the (formal) sentence into which it would be possible to translate the statement of the given theorem. To prove the theorem would then be to indicate how one could construct the formal proof whose existence is asserted by its statement. Mathematics as a whole would then become critical of an imaginary literature, possible but unrealised.

This hypothesis is certainly appealing at first glance. On the one hand, it restores the right of theorems to mean something, thereby re-establishing a line of communication between their statements and the intellectual act by which they are formulated. Moreover, the meaning it attributes to them is consistent with the intuitive content that many mathematicians already confer on them (insofar as this intuitive content is formal and not chosist). Finally, one need only read a modern mathematical work to be convinced that many of the proofs actually given are in the form of indications about the structure of a hypothetical, ungiven proof.

The hypothesis we have just presented can lead to curious speculations. For example, one may wonder what status should be given to a statement for which it could be demonstrated that a proof exists, without the proof of this fact allowing one to actually construct a proof of this statement. Such a statement cannot be considered (as long as no proof has actually been given) as a theorem of formal mathematics; nevertheless, mathematical opinion would probably be quite divided on the question of whether such a statement should be considered true.

But let us leave these considerations, which are somewhat untimely, aside for now, and return to examining the hypothesis we imagined above. Although we believe this hypothesis contains some truth, we do not think that it alone is sufficient to account for all existing mathematics. In particular, it fails to shed light on the nature of *calculations* (algebraic or otherwise) which, like the statements of theorems, form part of real mathematics. A calculation consists of a sequence of formulas written on paper which do not claim to *say* anything, but only to *be* what they are. It can be said that every calculation is a piece of formal mathematics inserted into the middle of real mathematics. It could be argued, if one wanted at all costs to save the hypothesis formulated above, that written formulas are only there as nouns designating assemblages of signs of formal mathematics; one could even add to corroborate this opinion that written formulas contain abbreviated signs, and that these signs cannot have the value of raw existents since they refer to meanings. But what ruins this position is that it in no way accounts for the attitude that the mathematician takes towards a calculation. Whether the signs used in formulas are abbreviated or not only comes into play at stages prior (setting up equations) and subsequent (interpreting results) to the actual calculation. But the mathematician who calculates is not a man who thinks, that is, who establishes relationships between symbols and meanings; he is a machine that performs certain material operations according to prescribed rules; the symbols used in his formulas have no other value for him - as long as he is calculating - than their very form and their arrangement on the page. It cannot be denied that calculations play an important role in mathematics: first and foremost, algebraic calculation, the oldest of all; set theory calculus with its formal operations on the symbols $\in$, $\subset$, $\supset$; approximate calculus, with its formal play on the symbols 0 and o; all "symbolic" methods, such as those of invariant theory, projective geometry, or Heaviside calculus; and even, finally, a kind of verbal calculus; insofar as long practice of a certain type of reasoning allows the mathematician to construct proofs in the same way that he calculates algebraically, by combining statements without referring to their meaning.

It therefore seems futile to try to fit the whole of mathematics into a single concept. Neither pure calculation nor pure thought, it comprises a part of calculations to which thought then gives various interpretations, and a part of thought that can occasionally materialise in calculations.

This dual aspect closely echoes M. Blanchot's description of literature (in the article cited above): "Literature is language that becomes ambiguity," he writes; and again:

> "Not only can every moment of language become ambiguous and say something other than what it says, but the general meaning of language is uncertain, and we do not know whether it expresses or represents, whether it is a thing or signifies it; ... whether it is transparent because of the lack of meaning in what it says or clear because of the accuracy with which it says it".

There are two different conceptions of mathematics, one as a pure verbal game, the other as a meaningful description of objects (real or imaginary), but there is only one mathematics, participating in both conceptions. Similarly, M. Blanchot writes:

> "Literature is divided between these two tendencies. The difficulty is that, although seemingly irreconcilable, they do not lead to distinct works or goals; and that art which claims to follow one slope is already on the other side. The first slope is that of meaningful prose... But, without leaving this side of language, there comes a moment when art perceives the dishonesty of everyday speech and departs from it... On the other slope... gather those we call poets. Why? Because they are interested in the reality of language, because they are not interested in the world, but in what things and beings would be like if there were no world...".

So poets are on the same side as calculators ! But it is true that Plato already brought them together in a common aversion. Here is how M. Blanchot speaks of a literary work:

> "Real words and an imaginary story, a world where everything that happens is borrowed from reality, and this world is inaccessible; ... so pure nothingness? But the book is there, you can touch it, the words can be read, they cannot be changed...".

His description applies word for word to a mathematical work.

The dual aspect—formal and meaningful—of literature is also the subject of Mr. Paulhan's reflections in his book *Les fleurs de Tarbes*[5], and here again the analogy continues with a situation that

[5] *Translator's note*: Jean Paulhan, *Les Fleurs de Tarbes*, Gallimard, 1936.

mathematicians know well. M. Paulhan gives the name “Terror” to the resolutely hostile attitude towards the art of rhetoric and language in general that literary criticism seems to have adopted since the 19th century. Terror is obsessed with the danger of seeing thought crushed by words and meaning lost in the automatic play of clichés and platitudes. Is this not the exact counterpart of the criticism often levelled at a certain type of mathematics: that it drowns thought in a sea of calculations and allows formalism to exceed the limits of the domain in which it is valid? Nineteenth-century mathematicians who returned to pure geometry despite the existence of analytical geometry, or those who focused on directly studying functions whose somewhat “teratological” nature prevented the application of infinitesimal calculus, were the protagonists of a reactionary movement similar to that of Romantic terror, a return to a direct experience of things. This tendency, which initially applied to not excluding the bizarre or the extraordinary as such, soon led to an excessive search for them for their own sake; in mathematics, the Polish school, and in literature, the prejudice that every novel hero must be distinguished by some absolutely new monstrosity. M. Paulhan proposes, as a remedy for excessive terror, the invention of a rhetoric whose clichés, recognised as such, serve as a vehicle for the movement of thought instead of stumbling over the (after all purely verbal) problem of avoiding clichés. Contemporary mathematics, for its part, seems to be losing interest in teratological counter-examples; the fact that a function may have no derivative anywhere, and other facts of the same order, quickly blunt the edge that may have caused a moment of delightful scandal. Rather than exhausting themselves in the search for increasingly monstrous deformities, contemporary mathematicians are turning instead to what could be described as the construction of a rhetoric, namely the development of new formalisms, modes of calculation strictly controlled by thought and designed not to dominate it but, on the contrary, to free it from part of its task.

We will not take sides in this work in favour of any of the conceptions that can be formed of the foundation and nature of mathematics; we believe we have shown in the preceding pages that none of these conceptions can be sufficient in itself, and that each one turns into its opposite if pushed far enough. “Everyone understands that literature cannot be divided and that to choose one’s place in it precisely, to convince oneself that one is exactly where one wanted to be, is to expose oneself to the greatest confusion, for literature has already insidiously moved you from one side to the other, changed you into what you were not," writes M. Blanchot; and this also applies to mathematics. To believe that perfect mathematics is possible, that absolute certainty can be achieved, is to court the confusion and despair that are the lot of the architects of proud Babels.

More modestly, we will try to do useful work by following the advice that M. Paulhan gives to writers, which is to take enough interest in language to make it an effective agent of thought. We believe that we have not wasted our time if we have truly made a number of "commonplaces" common; that is to say, if we have shown how mathematicians can benefit from a number of concepts whose highly abstract nature initially frightens the mind (such as, in algebra, the law of composition in general, or, in analysis, topological space, to cite just two examples). The first part of "Elements of Mathematics" will be devoted to the creation of these linguistic tools; the main aim will be to make them as easy to use as possible, so that they can be employed directly without having to retrace steps already taken. Ideally, by the end of each "book" of our work, the reader will be as familiar with the mathematical "structure" dealt with in that book as they are with the methods of elementary algebraic calculation; so that they would not hesitate to use an algebraically closed field or a compact space any more than a candidate for the grandes écoles would bat an eyelid at a first-degree equation. The sections following the first will deal with most of the major theories of classical and modern mathematics; we hope that reading them will convince you of the benefits of the work we have done in the first section to make the subject more accessible.